r-modes in superfluid neutron stars

Lee Lindblom

Theoretical Astrophysics 130-33, California Institute of Technology, Pasadena, California 91125

Gregory Mendell

Department of Physics and Astronomy, University of Wyoming, Laramie, Wyoming 82071

The analogues of *r*-modes in superfluid neutron stars are studied here. These modes, which are governed primarily by the Coriolis force, are identical to their ordinary-fluid counterparts at the lowest order in the small angular-velocity expansion used here. The equations that determine the next order terms are derived and solved numerically for fairly realistic superfluid neutron-star models. The damping of these modes by superfluid ''mutual friction'' (which vanishes at the lowest order in this expansion) is found to have a characteristic time scale of about 10^4 s for the $m=2$ *r*-mode in a ''typical'' superfluid neutron-star model. This time scale is far too long to allow mutual friction to suppress the recently discovered gravitational radiation driven instability in the *r*-modes. However, the strength of the mutual friction damping depends very sensitively on the details of the neutron-star core superfluid. A small fraction of the presently acceptable range of superfluid models have characteristic mutual friction damping times that are short enough (i.e., shorter than about 5 s) to suppress the gravitational radiation driven instability completely.

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I. INTRODUCTION

Recently Andersson [1] and Friedman and Morsink [2] showed that the *r*-modes in *all* rotating stars would be driven unstable by the emission of gravitational radiation in the absence of internal fluid dissipation. Subsequent analysis by Lindblom, Owen, and Morsink [3] and then by Andersson, Kokkotas, and Schutz [4] showed that internal fluid dissipation in hot young neutron stars is insufficient to suppress this gravitational radiation driven instability. Thus neutron stars that are formed rapidly rotating are expected to spin down within about one year to a relatively small angular velocity (about 5–10 % of the maximum) by the emission of gravitational radiation. Owen *et al.* [5] constructed rough models of this spindown process, and concluded that the gravitational radiation from these spindown events might be observable by the second-generation Laser Interferometer Gravitational-Wave Observatory (LIGO) gravitational-wave detectors.

The purpose of this paper is to investigate the behavior of this gravitational-radiation instability in the *r*-modes of older colder neutron stars [6–8]. Here the physics is more complicated and there are interesting observational constraints. The existence of the two 1.6 ms pulsars [9], and numerous examples of somewhat more slowly rotating neutron stars in low mass x-ray binaries (LMXBs) [10] show that some neutron stars are in fact rapidly rotating and stable. Unfortunately uncertainty in the neutron-star equation of state means that the minimum rotation periods for neutron stars are not presently well known, and so we cannot say exactly how rapid (in a dynamically meaningful sense) these rotations really are. Values for the minimum rotation periods (when mass shedding first occurs) of 1.4 $M_\odot$ models range from 0.5 ms to about 1.4 ms, depending on the equation of state [11]. Any value in this range, however, implies that the 1.6 ms pulsars are rotating much more rapidly than is consistent with our present understanding of the gravitational wave spindown due to the *r*-mode instability in hot young neutron

stars. It is widely believed, however, that these 1.6 ms objects are old cold recycled pulsars [12], having been spun up by accretion long after their initial cooldown. These neutron stars are expected to have superfluid cores, and hence the fluid dynamics and dissipation mechanisms that govern their *r*-modes are entirely different from those studied to date [1–5]. The purpose of this paper is to develop the tools needed to study the *r*-modes in superfluid neutron stars. The challenge is to understand how the *r*-mode instability is suppressed in the 1.6 ms pulsars in particular, and the more numerous 3 ms objects in LMXBs more generally.

The superfluid dissipation mechanism called ''mutual friction'' seems a likely candidate to provide the needed stability for the *r*-modes in old cold neutron stars. Mutual friction arises from the scattering of electrons off the magnetic fields entrapped in the cores of the superfluid neutron vortices [13,14], and is known to play an important role in other aspects of the dynamics of superfluid neutron stars. Lindblom and Mendell [15] show, for example, that mutual friction completely suppresses the gravitational radiation driven instability in the *f*-modes of rotating neutron stars. Our results here for the *r*-modes present a more ambiguous picture. We find in Sec. VI that the characteristic damping time for the *r*-modes due to mutual friction is about 10^4 s, for a typical model of the neutron-star core superfluid. This time scale is far too long to have any appreciable effect on the *r*-mode instability in these stars. However, we also find that the mutual-friction damping time is extremely sensitive to the parameters that define the core superfluid. Within the presently acceptable range of the parameters examined here, about 1% have mutual friction damping times so short (i.e., shorter than about 5 s) that the *r*-mode instability is suppressed completely. A somewhat larger fraction of these parameters in the acceptable range, about 3%, have damping times short enough (i.e., shorter than about 58 s) that mutual friction suppresses the instability in some sufficiently warm

and sufficiently slowly rotating neutron stars. Thus we conclude that an appropriately fine-tuned superfluid dynamics could provide the needed stability for the r-modes in old cold neutron stars through mutual friction. However given the small fraction of superfluid models that provide the needed stability, other damping mechanisms need to be considered (e.g., solid crust effects, strange quark matter, magnetic fields, . . .).

Regular shear viscosity is another mechanism that could play a role in damping the r-modes of superfluid neutron stars. We analyze this damping mechanism for the superfluid r-modes in Sec. VI and find a characteristic damping time of about $10^8 (T/10^9 \text{ K})^2$ s, where T is the temperature of the neutron-star core. This time scale is short enough to suppress the r-mode instability in stars cooler than about 10^6 K. Neutron stars are spun up in the usual picture during an LMXB phase, in which the core temperature is expected to exceed 10^8 K [16]. This temperature is too hot to allow shear viscosity to provide the needed stability even for the 3 ms neutron stars observed in these systems. Levin [8] has shown that when a neutron star is spun up to the point where stability of the r-modes is lost, the star heats up and then spins down to a very small angular velocity in a few months by emitting gravitational radiation. Thus, some robust internal fluid dissipation mechanism must be identified to explain the stability of the r-modes in the observed LMXB systems. If mutual friction is the only mechanism capable of providing the needed stability, then this fact would place interesting constraints on the parameters of the neutron-star core superfluid. An alternate possibility in the case of the 1.6 ms pulsars would be a mechanism for spinning up these stars without raising their core temperatures above about 10^7 K (e.g., by accretion at very low rates).

In Sec. II we review the basic hydrodynamics of neutron-star core superfluids. We outline in Sec. III the derivation of the equations that govern the normal modes of a superfluid neutron star from this hydrodynamic theory. In Sec. IV we take the small angular velocity expansions of these equations that are needed to study the r-modes. In Sec. V we present our numerical solutions for the r-modes of rotating superfluid neutron stars, up to the second order in the small angular velocity expansion. The effects of superfluid mutual friction and shear viscosity on these r-modes are evaluated in Sec. VI. The equations that determine the superfluid pulsations are expressed in spherical coordinates in the Appendix.

II. SUPERFLUID HYDRODYNAMICS IN NEUTRON-STAR MATTER

When the core temperature of a neutron star drops below about 10^9 K, a phase transition to a superconducting-superfluid state is expected to occur [17–20]. The neutrons in the core are expected to form $^3P_2$ Cooper pairs and the protons to form $^1S_0$ pairs. The purpose of this section is to review briefly the equations that describe the behavior of this complicated superconducting-superfluid mixture on the macroscopic scales needed here to describe the r-modes.

Let $\vec{v}_n$ denote the velocity of the neutron superfluid, and $\vec{v}_p$ the velocity of the proton superfluid. On small scales

these superfluid velocities, $\vec{v}_n$ and $\vec{v}_p$, are related by the London equations to the phases of the complex order parameters, S_n and S_p, that describe the neutron and proton condensates; thus, $\vec{v}_n = (\hbar/2m_n)\vec{\nabla} S_n$ and $\vec{v}_p = (\hbar/2m_p)\vec{\nabla} S_p - (e/m_p c)\vec{A}$, where $\vec{A}$ is the electromagnetic vector potential. These equations imply that vorticity and magnetic fields in this material are confined to vortices and flux tubes of microscopic dimension. In a typical neutron star the spacing between these neutron vortices is expected to be of order 10^{-3} cm, while the spacing between magnetic flux tubes is expected to be about 10^{-10} cm [19]. Our interest here is the very large-scale motions of this material associated with the low-order r-modes. Thus, it is appropriate to consider all physical quantities, such as $\vec{v}_n$ and $\vec{v}_p$, to be averaged over many vortices. The procedure for making this average is described more fully by Bekarevich and Khalatnikov [21], Baym and Chandler [22], Sonin [23], Mendell and Lindblom [24], and Mendell [25,26]. Throughout the remainder of this paper, all quantities are assumed to be so averaged.

One of the interesting and unusual features of the neutron-star core superfluid is the so called ''drag effect'' [27,13,19] or ''entrainment effect'' [28]. This effect is caused by the fact that the conserved particle currents are not simply proportional to the superfluid velocities. Instead these conserved currents are linear combinations of $\vec{v}_n$ and $\vec{v}_p$: $\rho_{nn}\vec{v}_n + \rho_{np}\vec{v}_p$ for the neutron current, and $\rho_{pp}\vec{v}_p + \rho_{np}\vec{v}_n$ for the proton current. Thus a given neutron superfluid flow $\vec{v}_n$ is accompanied by a certain (small) current of protons, and vice versa. The mass-density matrix elements ρ_{nn}, ρ_{pp}, and ρ_{np} are determined by the micro-physics of the many-body strong interactions that occur between the neutrons and protons. This entrainment effect plays a crucial role in mutual friction (perhaps the most important dissipation mechanism in this material) which we discuss in more detail in Sec. VI. Unfortunately these mass-matrix elements are not well determined at the present time. These quantities are constrained by Galilean invariance: $\rho_{nn} = \rho_n - \rho_{np}$ and $\rho_{pp} = \rho_p - \rho_{np}$, where ρ_n and ρ_p are the neutron and proton mass densities. But, the independent element ρ_{np} must be determined directly from the micro-physics. We find it convenient to re-express ρ_{np} in terms of the dimensionless entrainment parameter ϵ:

$$\rho_{np} = -\epsilon \rho_n. \tag{2.1}$$

Borumand, Joynt, and Kluźniak [29] estimate that $\epsilon \approx 0.04$, and that its value is known at present only to within about a factor of two. Given this uncertainty we explore the properties of the r-modes over the expected range of superfluid models with $0.02 \leq \epsilon \leq 0.06$.

The material in the core of a neutron star is a complicated mixture of neutrons, protons, electrons, muons, etc. While the general equations that describe the dynamics of this kind of charged superconducting-superfluid mixture have been studied [24], these general equations are considerably more complicated than are needed here. Our present interest is the dynamics of the superfluid analogues of the r-modes; thus we are interested in dynamics having length scales compa-

rable to the size, and time scales comparable to the rotation period, of the star. Under these conditions the dynamics of the core superfluid material simplifies considerably. On time scales longer than the plasma time scale (about 10^{-21} s) and the cyclotron time scale (about 10^{-15} s), for example, this material is well described by the magnetohydrodynamic limit of the exact equations [26]. In this limit the electrons and muons maintain exact charge neutrality with the protons. Similarly the electrons and muons are forced by scattering to move together as a single fluid on time scales longer than about 10^{-9} s [19]. Further, for dynamics on the time scales of interest here, the bulk electrical currents are extremely small [26]. Thus it is appropriate to simplify further and require the charged species to move together without generating any electrical current. The dynamical degrees of freedom of this material are reduced therefore to a pair of velocity vector fields—one for the neutrons and one for the protons—and a corresponding pair of thermodynamic scalar densities.

In general there are forces in the complete dynamical equations (even for this reduced system) that describe the interactions between the smoothed superfluid flow and the sheaf of vortices [24]. These additional forces are negligible for fluid motions with time scales comparable to the *r*-modes [25,26], and we neglect them here. And finally, in general the dynamics would also include a ''normal'' component of this superfluid material; however, again we simplify by assuming that the temperature is well below the superfluid transition and ignore these additional dynamical degrees of freedom.

Our study is directed toward an exploration of the superfluid analogues of the *r*-modes. We are primarily interested therefore in examining the equations that describe the evolution of small departures from a uniformly rotating equilibrium neutron star. The dynamics of the neutron-star core superfluid is described by two velocity vectors and two thermodynamic scalar fields. It will be convenient to express the equations for these velocity fields in terms of $\delta\vec{v}$ and $\delta\vec{w}$: the average and relative velocities of the core superfluids. These quantities are defined as

$$\rho\,\delta\vec{v} = \rho_n\delta\vec{v}_n + \rho_p\delta\vec{v}_p \qquad (2.2)$$

and

$$\delta\vec{w} = \delta\vec{v}_p - \delta\vec{v}_n. \qquad (2.3)$$

We use the prefix δ to denote a small (Eulerian) perturbation away from the equilibrium value of a quantity; while, quantities without prefix, such as $\rho = \rho_n + \rho_p$, denote the equilibrium values. The superfluid velocity fields $\delta\vec{v}_n$ and $\delta\vec{v}_p$ are easily determined from Eqs. (2.2) and (2.3) once $\delta\vec{v}$ and $\delta\vec{w}$ are known. Similarly the velocity field of the electrons, $\delta\vec{v}_e$, can be expressed in terms of these quantities from the condition that there is no electrical current [25]. The electrical current of protons is proportional to the mass current, which is $\rho_{pp}\vec{v}_p + \rho_{np}\vec{v}_n$ because of the entrainment effect; thus the perturbed electron velocity is given by

$$\delta\vec{v}_e = \frac{\rho_{pp}}{\rho_p}\,\delta\vec{v}_p + \frac{\rho_{np}}{\rho_p}\,\delta\vec{v}_n = \delta\vec{v} + \frac{\rho_n}{\rho}\left(1 + \epsilon\,\frac{\rho}{\rho_p}\right)\delta\vec{w}. \qquad (2.4)$$

We note that for simplicity in these equations we have ignored terms of order m_e/m_p, the ratio of electron to proton mass. And for simplicity in this discussion, we have also ignored the presence of muons. A more complete discussion including the contributions of the muons is given by Mendell [25].

The equations for the velocity fields $\delta\vec{v}$ and $\delta\vec{w}$ are obtained by perturbing the full system of superconducting-superfluid evolution equations, subject to the assumptions described above. These equations, when expressed in terms of $\delta\vec{v}$ and $\delta\vec{w}$, have the remarkably simple forms [30]:

$$\partial_t\delta v^a + v^b\nabla_b\delta v^a + \delta v^b\nabla_b v^a = -\nabla^a\delta U + \frac{1}{\rho^2}\left(\frac{\partial\rho}{\partial\beta}\right)_p\delta\beta\,\nabla^a p, \qquad (2.5)$$

$$\partial_t\delta w^a + v^b\nabla_b\delta w^a + (2\gamma-1)\delta w^b\nabla_b v^a = -\nabla^a\delta\beta. \qquad (2.6)$$

The perturbed scalar δU that appears on the right-hand side of Eq. (2.5) is defined by

$$\delta U = \frac{\delta p}{\rho} - \delta\Phi, \qquad (2.7)$$

where δp is the perturbed pressure, and $\delta\Phi$ the perturbed gravitational potential. The potential $\delta\beta$ that appears on the right-hand sides of Eqs. (2.5) and (2.6) measures the degree to which the perturbed fluid departs from β equilibrium. The thermodynamic function β is related to the chemical potentials (per unit mass) of the neutrons μ_n, protons μ_p, and electrons μ_e by

$$\beta = \mu_p - \mu_n + \frac{m_e}{m_p}\mu_e. \qquad (2.8)$$

The quantity β vanishes in the equilibrium state. (For simplicity we again neglect terms of order m_e/m_p.) Finally, the velocity field v^a in Eqs. (2.5) and (2.6) represents the uniform rotation of the equilibrium star, and the dimensionless quantity γ that appears in Eq. (2.6) is related to the determinant of the superfluid mass-density matrix: $\gamma = (\rho_{nn}\rho_{pp} - \rho_{np}^2)/\rho_n\rho_p$.

The three perturbed scalar fields, δU, $\delta\Phi$, and $\delta\beta$, that appear on the right-hand sides of Eqs. (2.5) and (2.6) can be used to determine all of the other scalars of interest in this problem; for example

$$\delta\rho = \rho\left(\frac{\partial\rho}{\partial p}\right)_\beta(\delta U + \delta\Phi) + \left(\frac{\partial\rho}{\partial\beta}\right)_p\delta\beta, \qquad (2.9)$$

and $\delta\rho_p$ (and thence $\delta\rho_n$ as well) can be determined from

$$\delta\beta = \left[\left(\frac{\partial\mu_p}{\partial\rho_n}\right)_{\rho_p} - \left(\frac{\partial\mu_n}{\partial\rho_n}\right)_{\rho_p}\right](\delta\rho - \delta\rho_p)$$

$$+ \left[\left(\frac{\partial\mu_p}{\partial\rho_p}\right)_{\rho_n} - \left(\frac{\partial\mu_n}{\partial\rho_p}\right)_{\rho_n} + \frac{m_e^2}{m_p^2}\frac{d\mu_e}{d\rho_e}\right]\delta\rho_p. \quad (2.10)$$

It is straightforward then to transform the mass conservation laws (for neutrons and protons) into forms depending only on δU, $\delta\Phi$, and $\delta\beta$ [30]:

$$\partial_t\delta\rho + v^a\nabla_a\delta\rho + \nabla_a(\rho\,\delta v^a) = 0, \quad (2.11)$$

$$(\partial_t + v^a\nabla_a)\left[\left(\frac{\partial\rho}{\partial\beta}\right)_p(\delta U + \delta\Phi) + \frac{\rho_n^2}{\rho}\frac{\partial}{\partial\beta}\left(\frac{\rho_p}{\rho_n}\right)_p\delta\beta\right]$$

$$+ \frac{1}{\rho}\left(\frac{\partial\rho}{\partial\beta}\right)_p\delta v^a\nabla_a p + \nabla_a(\tilde\rho\,\delta w^a) = 0. \quad (2.12)$$

The quantity $\tilde\rho$ that appears in Eq. (2.12) is defined as $\tilde\rho = (\rho_{nn}\rho_{pp} - \rho_{np}^2)/\rho = \rho_n\rho_p\gamma/\rho$. These equations, (2.5), (2.6), (2.11), and (2.12), together with the perturbed gravitational potential equation,

$$\nabla^a\nabla_a\delta\Phi = -4\pi G\,\delta\rho, \quad (2.13)$$

determine the evolution of the material in the superfluid core of a neutron star in the long length scale, long time scale, and low temperature approximation of interest to us here.

III. OSCILLATIONS OF SUPERFLUID NEUTRON STARS

A superfluid neutron star is a reasonably complicated structure consisting of a superfluid core surrounded by a solid crust, and probably a liquid ocean above that. For the purposes of our analysis here we use a simplified and idealized representation of this structure. We consider a neutron-star model that consists of a superfluid core (where $\rho > \rho_s$), surrounded by an ordinary matter envelope (where $\rho < \rho_s$). For simplicity we treat the material in this envelope as a perfect fluid. The dynamics of the material in the core is described by Eqs. (2.5), (2.6), (2.11), (2.12), and (2.13). And similarly the material in our idealized envelope is described by Euler's equation, Eq. (2.5) with $\delta\beta = 0$, and Eqs. (2.11) and (2.13). In this section we show how the modes of such a rotating superfluid stellar model can be described completely

in terms of the three scalar potentials δU, $\delta\beta$, and $\delta\Phi$ [30]. And we derive the equations that determine these potentials, together with the appropriate boundary conditions.

We assume here that the time dependence of the perturbation is $e^{i\omega t}$ and that its azimuthal angular dependence is $e^{im\varphi}$, where ω is the frequency of the mode and m is an integer. The superfluid versions of the Euler equation, Eqs. (2.5) and (2.6), determine the velocities δv^a and δw^a in terms of the scalars δU and $\delta\beta$ much as they do in the ordinary fluid case [31]. Given the temporal and angular dependence assumed here, Eqs. (2.5) and (2.6) become linear algebraic equations for δv^a and δw^a which can be solved directly:

$$\delta v^a = iQ^{ab}\left[\nabla_b\delta U - \frac{1}{\rho^2}\left(\frac{\partial\rho}{\partial\beta}\right)_p\delta\beta\nabla_b p\right], \quad (3.1)$$

$$\delta w^a = i\tilde Q^{ab}\nabla_b\delta\beta. \quad (3.2)$$

In these equations Q^{ab} and $\tilde Q^{ab}$ are tensors that depend on the frequency of the mode ω, and the angular velocity of the equilibrium star Ω. These tensors are given by

$$Q^{ab} = \frac{1}{(\omega + m\Omega)^2 - 4\Omega^2}\left[(\omega + m\Omega)\delta^{ab}\right.$$

$$\left. - \frac{4\Omega^2 z^a z^b}{\omega + m\Omega} - 2i\Omega\nabla^a\varphi^b\right], \quad (3.3)$$

$$\tilde Q^{ab} = \frac{1}{(\omega + m\Omega)^2 - 4\gamma^2\Omega^2}\left[(\omega + m\Omega)\delta^{ab}\right.$$

$$\left. - \frac{4\gamma^2\Omega^2 z^a z^b}{\omega + m\Omega} - 2i\gamma\Omega\nabla^a\varphi^b\right]. \quad (3.4)$$

In Eqs. (3.3) and (3.4) Ω is the angular velocity of the equilibrium star; the unit vector z^a points along the rotation axis; φ^a is the vector field that generates rotations about the z^a axis; and δ^{ab} is the Euclidean metric tensor (the identity matrix in Cartesian coordinates).

The expressions for the velocity fields in Eqs. (3.1) and (3.2) can be substituted into the mass conservation laws, Eqs. (2.11) and (2.12), to obtain equations for the scalar fields alone. In general, the potentials δU, $\delta\beta$, and $\delta\Phi$ are solutions then of the following system of partial differential equations [30]:

$$\nabla_a(\rho Q^{ab}\nabla_b\delta U) + (\omega + m\Omega)\rho\left(\frac{\partial\rho}{\partial p}\right)_\beta\delta U = \nabla_a\left[\frac{1}{\rho}\left(\frac{\partial\rho}{\partial\beta}\right)_p\delta\beta\,Q^{ab}\nabla_b p\right] - (\omega + m\Omega)\left[\left(\frac{\partial\rho}{\partial\beta}\right)_p\delta\beta + \rho\left(\frac{\partial\rho}{\partial p}\right)_\beta\delta\Phi\right], \quad (3.5)$$

$$\nabla_a(\tilde\rho\tilde Q^{ab}\nabla_b\delta\beta) + (\omega + m\Omega)\frac{\rho_n^2}{\rho}\frac{\partial}{\partial\beta}\left(\frac{\rho_p}{\rho_n}\right)_p\delta\beta - \frac{1}{\rho^3}\left(\frac{\partial\rho}{\partial\beta}\right)_p^2 Q^{ab}\nabla_a p\nabla_b p\,\delta\beta$$

$$= -\left(\frac{\partial\rho}{\partial\beta}\right)_p\left[\frac{1}{\rho}Q^{ab}\nabla_a p\nabla_b\delta U + (\omega + m\Omega)(\delta U + \delta\Phi)\right], \quad (3.6)$$

$$\nabla^a \nabla_a \delta\Phi + 4\pi G \rho \left(\frac{\partial\rho}{\partial p} \right)_\beta \delta\Phi$$

$$= -4\pi G \rho \left(\frac{\partial\rho}{\partial p} \right)_\beta \delta U - 4\pi G \left(\frac{\partial\rho}{\partial\beta} \right)_p \delta\beta. \quad (3.7)$$

The functions δU, $\delta\beta$, and $\delta\Phi$ are also subject to appropriate boundary conditions at the interface between the superfluid core and the ordinary-fluid envelope, at the surface of the star, and at infinity. First, we consider the boundary at the interface between the superfluid core and the ordinary-fluid envelope of the star. Mass and momentum conservation across this boundary place a number of constraints on the continuity of these functions [30]. In particular these conditions require that the functions δU and $\delta\Phi$ be continuous there. In addition, $\nabla_a \delta\Phi$ must be continuous, while $\nabla_a \delta U$ must have a discontinuity that is prescribed by

$$n^a [\nabla_a \delta U]_s - \frac{1}{\rho^2} \left(\frac{\partial\rho}{\partial\beta} \right)_p n^a \nabla_a p [\delta\beta]_s = n^a [\nabla_a \delta U]_o. \quad (3.8)$$

The subscripts s and o in Eq. (3.8) denote that the quantities are to be evaluated as limits from the superfluid or the ordinary-fluid side of the boundary, respectively, and n^a denotes the outward directed unit normal to the boundary surface. The function $\delta\beta$, which is of interest to us only within the superfluid core, must satisfy the condition

$$n^a [\nabla_a \delta\beta]_s - \frac{4\gamma^2 \Omega^2 z^b n_b}{(\omega + m\Omega)^2} z^a [\nabla_a \delta\beta]_s + \frac{2m\gamma\Omega \varpi^b n_b}{(\omega + m\Omega)\varpi} [\delta\beta]_s$$
$$= 0, \quad (3.9)$$

on the boundary of the superfluid core. Here we use the notation ϖ for the cylindrical radial coordinate, and ϖ^a to denote the unit vector in the ϖ direction.

Next consider the boundary conditions on the outer surface of the star. The function δU must be constrained at this surface in such a way that the Lagrangian perturbation in the

pressure vanishes there: $\Delta p = 0$. This condition can be written in terms of the variables used here by noting that

$$\Delta p = \delta p + \frac{\delta v^a \nabla_a p}{i(\omega + m\Omega)}, \quad (3.10)$$

where δv^a is given in this region by Eq. (3.1) with $\delta\beta = 0$. Thus using Eqs. (2.7) and (3.1) this boundary condition can be written in terms of δU and $\delta\Phi$ as

$$0 = [\rho(\omega + m\Omega)(\delta U + \delta\Phi) + Q^{ab} \nabla_a p \nabla_b \delta U]_o. \quad (3.11)$$

Finally, the perturbed gravitational potential $\delta\Phi$ must fall off at infinity faster than $1/r$ in order that the mass of the perturbed star remain the same as that of the equilibrium star: $\lim_{r\to\infty} (r\delta\Phi) = 0$. In addition $\delta\Phi$ and its first derivative must be continuous at the surface of the star. The problem of finding the modes of ''uniformly'' rotating superfluid stars is reduced therefore to finding the solutions to Eqs. (3.5), (3.6), and (3.7) subject to the appropriate boundary conditions including in particular Eqs. (3.8), (3.9), and (3.11).

The equations for the potentials δU and $\delta\beta$, Eqs. (3.5) and (3.6), have complicated dependences on the frequency of the mode and the angular velocity of the star through Q^{ab} and $\tilde{Q}^{ab}$, as given in Eqs. (3.3) and (3.4). In the analysis that follows it will be necessary to have those dependences displayed more explicitly. Here we are interested in investigating the superfluid versions of r-modes. Such modes have frequencies that go to zero linearly as the angular velocity of the star vanishes. Thus, it will be useful to define the dimensionless frequency parameter κ:

$$\kappa\Omega = \omega + m\Omega. \quad (3.12)$$

The parameter κ remains finite in the zero angular-velocity limit for these modes. Using this parameter and the expressions for Q^{ab} and $\tilde{Q}^{ab}$ from Eqs. (3.3) and (3.4), we re-write Eqs. (3.5) and (3.6) to obtain the following equivalent forms:

$$\nabla_a [\rho(\kappa^2 \delta^{ab} - 4z^a z^b) \nabla_b \delta U] + \frac{2m\kappa}{\varpi} \varpi^a \nabla_a \rho \; \delta U = -\kappa^2(\kappa^2 - 4)\Omega^2 \left[\rho \left(\frac{\partial\rho}{\partial p} \right)_\beta (\delta U + \delta\Phi) + \left(\frac{\partial\rho}{\partial\beta} \right)_p \delta\beta \right]$$

$$+ \nabla_a \left[\frac{1}{\rho} \left(\frac{\partial\rho}{\partial\beta} \right)_p \delta\beta \, (\kappa^2 \delta^{ab} - 4z^a z^b) \nabla_b p \right] - \frac{2m\kappa}{\rho\varpi} \left(\frac{\partial\rho}{\partial\beta} \right)_p \delta\beta \, \varpi^a \nabla_a p, \quad (3.13)$$

$$\nabla_a \left[\frac{\tilde{\rho}}{\kappa^2 - 4\gamma^2} (\kappa^2 \delta^{ab} - 4\gamma^2 z^a z^b) \nabla_b \delta\beta \right] + \left[\frac{2m\kappa}{\varpi} \varpi^a \nabla_a \left(\frac{\gamma\tilde{\rho}}{\kappa^2 - 4\gamma^2} \right) + \kappa^2 \Omega^2 \frac{\rho_n^2}{\rho} \frac{\partial}{\partial\beta} \left(\frac{\rho_p}{\rho_n} \right)_p \right] \delta\beta$$

$$- \frac{1}{\rho^3(\kappa^2 - 4)} \left(\frac{\partial\rho}{\partial\beta} \right)_p^2 (\kappa^2 \delta^{ab} - 4z^a z^b) \nabla_a p \nabla_b p \; \delta\beta$$

$$= - \left(\frac{\partial\rho}{\partial\beta} \right)_p \left[\frac{\kappa^2 \delta^{ab} - 4z^a z^b}{\rho(\kappa^2 - 4)} \nabla_a p \nabla_b \delta U + \frac{2m\kappa}{\rho(\kappa^2 - 4)\varpi} \varpi^a \nabla_a p \; \delta U + \kappa^2 \Omega^2 (\delta U + \delta\Phi) \right]. \quad (3.14)$$

The boundary conditions, Eqs. (3.9) and (3.11), are similarly transformed into the forms:

$$\kappa^2 n^a [\nabla_a \delta\beta]_s - 4\gamma^2 z^b n_b z^a [\nabla_a \delta\beta]_s + \frac{2m\kappa\gamma\varpi^b n_b}{\varpi}[\delta\beta]_s = 0. \tag{3.15}$$

$$\left[(\kappa^2\delta^{ab} - 4z^a z^b)\nabla_a h\nabla_b \delta U + \frac{2m\kappa}{\varpi}\varpi^a\nabla_a h \ \delta U + \kappa^2(\kappa^2 - 4)\Omega^2(\delta U + \delta\Phi)\right]_o = 0. \tag{3.16}$$

In Eq. (3.16) we have expressed the boundary condition in terms of the thermodynamic enthalpy, h, which is defined as

$$h(p) = \int_0^p \frac{dp'}{\rho(p')}. \tag{3.17}$$

The enthalpy is the appropriate thermodynamic function to use in Eq. (3.16) because its gradient, $\nabla_a h$, is well-defined and nonzero at the surface of the star.

IV. SUPERFLUID r-MODES IN THE SLOW ROTATION APPROXIMATION

The r-modes of rotating ordinary-fluid stars have traditionally been studied using a small angular-velocity expansion [32]. Our goal here is to perform a similar expansion for the superfluid generalizations of the r-modes. Thus, we seek solutions to Eqs. (3.13), (3.14), and (3.7) as power series in the angular velocity of the star.

We begin first with the structure of the equilibrium superfluid star. This structure is identical to its ordinary-fluid counterpart in the large-scale averaged-over-vortices hydrodynamics used here. We expand each of the equilibrium functions of interest:

$$\rho = \rho_0 + \rho_2 \frac{\Omega^2}{\pi G\bar{\rho}_0} + \mathcal{O}(\Omega^4), \tag{4.1}$$

$$p = p_0 + p_2 \frac{\Omega^2}{\pi G\bar{\rho}_0} + \mathcal{O}(\Omega^4), \tag{4.2}$$

$$h = h_0 + h_2 \frac{\Omega^2}{\pi G\bar{\rho}_0} + \mathcal{O}(\Omega^4). \tag{4.3}$$

The location of the surface of the star $R(\mu)$ is also expressed as such an expansion:

$$R = R_0 + R_2 \frac{\Omega^2}{\pi G\bar{\rho}_0} + \mathcal{O}(\Omega^4). \tag{4.4}$$

Here and throughout the remainder of this paper we use the subscripts 0 and 2 to denote the lowest- and the second-order terms respectively in these expansions; and we use r and $\mu = \cos\theta$ to denote the standard spherical coordinates. We also introduce here the angular velocity scale $\sqrt{\pi G\bar{\rho}_0}$, where $\bar{\rho}_0$ is the average density of the star in the non-rotating limit.

(Equilibrium neutron-star models do not exist for $\Omega \gtrsim \frac{2}{3}\sqrt{\pi G\bar{\rho}_0}$.) The techniques needed to evaluate the terms in these series for the equilibrium structure are identical to those described for example in Lindblom, Mendell, and Owen [33].

Next, we define expansions for the quantities that determine the perturbations of a superfluid star, δU, $\delta\beta$, $\delta\Phi$, and κ:

$$\delta U = R_0^2 \Omega^2 \left[\delta U_0 + \delta U_2 \frac{\Omega^2}{\pi G\bar{\rho}_0} + \mathcal{O}(\Omega^4)\right], \tag{4.5}$$

$$\delta\beta = R_0^2 \Omega^2 \left[\delta\beta_0 + \delta\beta_2 \frac{\Omega^2}{\pi G\bar{\rho}_0} + \mathcal{O}(\Omega^4)\right], \tag{4.6}$$

$$\delta\Phi = R_0^2 \Omega^2 \left[\delta\Phi_0 + \delta\Phi_2 \frac{\Omega^2}{\pi G\bar{\rho}_0} + \mathcal{O}(\Omega^4)\right], \tag{4.7}$$

$$\kappa = \kappa_0 + \kappa_2 \frac{\Omega^2}{\pi G\bar{\rho}_0} + \mathcal{O}(\Omega^4). \tag{4.8}$$

We have normalized the eigenfunctions using R_0, the radius of the star (in the nonrotating limit), and Ω, the angular velocity of the star. Using these expressions for the perturbations, together with those for the structure of the equilibrium star, it is straightforward to write down order-by-order the equations that determine the superfluid r-modes. This expansion is completely analogous to that given by Lindblom, Mendell, and Owen [33] for the ordinary-fluid modes.

It is straightforward to verify that the functions

$$\delta U_0 = \alpha\left(\frac{r}{R_0}\right)^{m+1} P_{m+1}^m(\mu)e^{im\varphi}, \tag{4.9}$$

$$\delta\beta_0 = 0, \tag{4.10}$$

with

$$\kappa_0 = \frac{2}{m+1}, \tag{4.11}$$

satisfy the lowest-order terms from the expansion of the pulsation Eqs. (3.13) and (3.14). Inspection of the equation for $\delta\beta$, Eq. (3.14), reveals that the right side is proportional to Ω^3: the first term on the right vanishes identically for δU_0 given by Eq. (4.9), while the lowest-order contribution to the

second term is proportional to Ω^3. Thus the function $\delta\beta_0 = 0$ satisfies Eq. (3.14) to lowest order.

With the lowest-order contribution to $\delta\beta$ vanishing, the lowest-order equations for δU and $\delta\Phi$ from Eqs. (3.13) and (3.7) reduce to the following,

$$\nabla_a[\rho_0(\kappa_0^2\delta^{ab}-4z^a z^b)\nabla_b\delta U_0]+\frac{2m\kappa_0}{\varpi}\varpi^a\nabla_a\rho_0\ \delta U_0=0,\tag{4.12}$$

$$\nabla^a\nabla_a\delta\Phi_0=-4\pi G\left(\frac{d\rho}{dh}\right)_0(\delta U_0+\delta\Phi_0).\tag{4.13}$$

The lowest-order boundary conditions at the superfluid ordinary-fluid interface, Eqs. (3.8) and (3.9), merely require that δU_0 and $\nabla_a\delta U_0$ are continuous there. The lowest-order contribution from the boundary condition for δU_0 at the surface of the star is

$$\left[(\kappa_0^2\delta^{ab}-4z^a z^b)\nabla_a h_0\nabla_b\delta U_0+\frac{2m\kappa_0}{\varpi}\varpi^a\nabla_a h_0\ \delta U_0\right]_{r=R_0}$$

$$=0.\tag{4.14}$$

These equations are identical to the lowest-order terms in the ordinary-fluid *r*-mode equations [33]. The function δU_0 together with κ_0 given in Eqs. (4.9) and (4.11) satisfy Eqs. (4.12) and (4.14) identically because these are in fact the lowest-order expressions for the classical *r*-modes as studied for example by Papaloizou and Pringle [32]. (However, they are expressed here in a form that was introduced more recently [34].) Thus to lowest order the superfluid *r*-modes are identical to their ordinary-fluid counterparts.

Continuing on to second order, the equations for the potentials are

$$\nabla_a[\rho_0(\kappa_0^2\delta^{ab}-4z^a z^b)\nabla_b\delta U_2]+\frac{2m\kappa_0}{\varpi}\varpi^a\nabla_a\rho_0\ \delta U_2+\nabla_a[\rho_2(\kappa_0^2\delta^{ab}-4z^a z^b)\nabla_b\delta U_0+2\kappa_0\kappa_2\rho_0\nabla^a\delta U_0]$$

$$+\frac{2m}{\varpi}\varpi^a(\kappa_2\nabla_a\rho_0+\kappa_0\nabla_a\rho_2)\delta U_0$$

$$=-\pi G\bar\rho_0\kappa_0^2(\kappa_0^2-4)\rho_0\left(\frac{\partial\rho}{\partial p}\right)_\beta(\delta U_0+\delta\Phi_0)+\nabla_a\left[\frac{1}{\rho_0}\left(\frac{\partial\rho}{\partial\beta}\right)_p\delta\beta_2(\kappa_0^2\delta^{ab}-4z^a z^b)\nabla_b p_0\right]$$

$$-\frac{2m\kappa_0}{\varpi}\frac{1}{\rho_0}\left(\frac{\partial\rho}{\partial\beta}\right)_p\delta\beta_2\varpi^a\nabla_a p_0,\tag{4.15}$$

$$\nabla_a\left[\frac{\tilde\rho_0}{\kappa_0^2-4\gamma_0^2}(\kappa_0^2\delta^{ab}-4\gamma_0^2 z^a z^b)\nabla_b\delta\beta_2\right]+\frac{2m\kappa_0}{\varpi}\varpi^a\nabla_a\left(\frac{\gamma_0\tilde\rho_0}{\kappa_0^2-4\gamma_0^2}\right)\delta\beta_2-\frac{1}{\rho_0^3(\kappa_0^2-4)}\left(\frac{\partial\rho}{\partial\beta}\right)_p^2(\kappa_0^2\delta^{ab}-4z^a z^b)\nabla_a p_0\nabla_b p_0\ \delta\beta_2$$

$$=-\frac{1}{\rho_0(\kappa_0^2-4)}\left(\frac{\partial\rho}{\partial\beta}\right)_p\left[(\kappa_0^2\delta^{ab}-4z^a z^b)\nabla_a p_0\nabla_b\delta U_2+\frac{2m\kappa_0}{\varpi}\varpi^a\nabla_a p_0\ \delta U_2+2\kappa_0\kappa_2\nabla^a p_0\nabla_a\delta U_0+\frac{2m\kappa_2}{\varpi}\varpi^a\nabla_a p_0\ \delta U_0\right.$$

$$\left.+\frac{2m\kappa_0}{\varpi}\varpi^a\nabla_a p_2\ \delta U_0+(\kappa_0^2\delta^{ab}-4z^a z^b)\nabla_a p_2\nabla_b\delta U_0\right]-\kappa_0^2\pi G\bar\rho_0\left(\frac{\partial\rho}{\partial\beta}\right)_p(\delta U_0+\delta\Phi_0),\tag{4.16}$$

$$\nabla^a\nabla_a\delta\Phi_2+4\pi G\rho_0\left(\frac{\partial\rho}{\partial p}\right)_\beta\delta\Phi_2=-4\pi G\rho_0\left(\frac{\partial\rho}{\partial p}\right)_\beta\delta U_2-4\pi G\left(\frac{\partial\rho}{\partial\beta}\right)_p\delta\beta_2-4\pi G\left(\frac{\partial\rho}{\partial p}\right)_\beta^{-1}\frac{\partial}{\partial p}\left[\rho_0\left(\frac{\partial\rho}{\partial p}\right)_\beta\right]_\beta\rho_2(\delta U_0+\delta\Phi_0).\tag{4.17}$$

It will be helpful to express Eqs. (4.15) and (4.16) in the following shorthand forms:

$$D(\delta U_2)=E(\delta\beta_2)+F,\tag{4.18}$$

$$\tilde D(\delta\beta_2)=\tilde E(\delta U_2)+\tilde F,\tag{4.19}$$

where D and $\tilde D$ are second-order partial differential operators, E and $\tilde E$ are first-order operators, and F and $\tilde F$ are functions that depend on the lowest-order eigenfunctions δU_0 and $\delta\Phi_0$.

We next consider the second-order boundary conditions. The boundary conditions that must be satisfied at the superfluid ordinary-fluid boundary, Eqs. (3.8) and (3.9), have the following second-order forms on the surface $r=R_s$:

$$\left[\frac{\partial\delta U_2}{\partial r}\right]_s - \frac{1}{\rho_0^2}\left(\frac{\partial\rho}{\partial\beta}\right)_p\frac{dp_0}{dr}[\delta\beta_2]_s = \left[\frac{\partial\delta U_2}{\partial r}\right]_o, \tag{4.20}$$

$$(\kappa_0^2 - 4\gamma_0^2\mu^2)\left[\frac{\partial\delta\beta_2}{\partial r}\right]_s - \frac{4\gamma_0^2}{R_s}\mu(1-\mu^2)\left[\frac{\partial\delta\beta_2}{\partial\mu}\right]_s + \frac{2m\kappa_0\gamma_0}{R_s}[\delta\beta_2]_s = 0. \tag{4.21}$$

The second-order boundary condition for the potential δU, Eq. (3.16), is identical to that derived by Lindblom, Mendell, and Owen [33] for the perfect-fluid case:

$$\left\{(\kappa_0^2\delta^{ab}-4z^az^b)\nabla_a h_0\nabla_b\delta U_2 + \frac{2m\kappa_0}{\varpi}\varpi^a\nabla_a h_0\delta U_2 + (\kappa_0^2\delta^{ab}-4z^az^b)\nabla_a h_2\nabla_b\delta U_0 + \frac{2m\kappa_0}{\varpi}\varpi^a\nabla_a h_2\delta U_0\right.$$

$$\left. + 2\kappa_0\kappa_2\nabla^a h_0\nabla_a\delta U_0 + \frac{2m\kappa_2}{\varpi}\varpi^a\nabla_a h_0\delta U_0 + \kappa_0^2(\kappa_0^2-4)\pi G\bar\rho_0(\delta U_0+\delta\Phi_0)\right\}_{r=R_0} = 0. \tag{4.22}$$

It will be useful in our numerical solution of the pulsation equations to have an expression for the second-order contribution to the frequency of the mode, κ_2, as integrals over the eigenfunctions. Such an expression can be obtained by multiplying Eq. (4.18) by δU_0^* and integrating over the interior of the star. Since δU_0^* is an element of the kernel of the operator D, this part of the integral reduces to a boundary integral. This boundary integral is nonvanishing because $\nabla_a\delta U_2$ is discontinuous at the superfluid boundary as a consequence of the boundary condition Eq. (4.20). The result of integrating the left-hand side of Eq. (4.18) can be expressed in the following way:

$$\int \delta U_0^* D(\delta U_2)d^3x = 2\pi\left[r^2\left(\frac{\partial\rho}{\partial\beta}\right)_p\frac{dh_0}{dr}\int_{-1}^1(\kappa_0^2-4\mu^2)\delta\beta_2\delta U_0^* d\mu\right]_{r=R_s}. \tag{4.23}$$

Combining this result with the more straightforward integrals of the right-hand side of Eq. (4.18), we obtain the following expression for the second-order change in the frequency of the superfluid r-mode:

$$\kappa_2\int\frac{1}{r}\frac{d\rho_0}{dr}|\delta U_0|^2 d^3x = \frac{6m}{(m+1)^2}\int\frac{\rho_{22}}{r^2}|\delta U_0|^2 d^3x + \frac{8\pi G\bar\rho_0 m}{(m+1)^4}\int\left(\frac{d\rho}{dh}\right)_0(\delta U_0+\delta\Phi_0)\delta U_0^* d^3x$$

$$+ \int\frac{(\kappa_0^2-4\mu^2)r}{2(m+2)}\frac{d}{dr}\left[\frac{1}{r}\frac{dh_0}{dr}\left(\frac{\partial\rho}{\partial\beta}\right)_p\right]\delta\beta_2\delta U_0^* d^3x + \int\frac{dh_0}{dr}\left(\frac{\partial\rho}{\partial\beta}\right)_p$$

$$\times\left[\frac{\kappa_0^2-4\mu^2}{2(m+2)}\frac{\partial\delta\beta_2}{\partial r}-\frac{2\mu(1-\mu^2)}{(m+2)r}\frac{\partial\delta\beta_2}{\partial\mu}\right]\delta U_0^* d^3x + \frac{3\kappa_0^2-2m\kappa_0-4}{2(m+2)}\int\frac{1}{r}\frac{dh_0}{dr}\left(\frac{\partial\rho}{\partial\beta}\right)_p\delta\beta_2\delta U_0^* d^3x$$

$$-\left[\pi r^2\frac{dh_0}{dr}\left(\frac{\partial\rho}{\partial\beta}\right)_p\int\frac{\kappa_0^2-4\mu^2}{m+2}\delta\beta_2\delta U_0^* d\mu\right]_{r=R_s}. \tag{4.24}$$

In Eq. (4.24) the integrations are to be performed over the interior of the superfluid core, $0\leq r\leq R_s$, for those integrals involving $\delta\beta_2$ and throughout the interior of the star, $0\leq r\leq R_0$, for those integrals that do not. Even though this expression for κ_2 depends on the second-order eigenfunction $\delta\beta_2$ it is nevertheless very useful in determining κ_2 numerically. This is due to the fact that the integrals involving $\delta\beta_2$ are typically quite small.

V. NUMERICAL SOLUTIONS FOR THE SUPERFLUID r-MODES

In order to solve the equations for the superfluid r-modes we must adopt a specific model for the equilibrium structure

of the neutron star, and explicit expressions for the various thermodynamic functions that appear in the equations. For the equilibrium structure of the neutron star, we use the simple polytropic equation of state: $p=K\rho^2$, with K chosen so that a $1.4M_\odot$ model has a radius of $R_0=12.533$ km. We choose this simple model because our method of solving Eq. (4.18) to determine δU_2 seems to be rather unstable for more realistic equations of state. We also use this simple equilibrium equation of state to evaluate the thermodynamic derivative $(\partial\rho/\partial p)_\beta=\rho/2p$ that appears in the pulsation equations. We adopt the fiducial value $\rho_s=2.8\times10^{14}$ g/cm^3 for the superfluid transition density. However, we examine the sensitivity of our results to variations in this parameter.

In order to evaluate the other thermodynamic properties

of neutron star matter needed in the pulsation equations, we use the recent semi-empirical equation of state, A18+δv +UIX, of Akmal, Pandharipande, and Ravenhall [35]. The derivative $(\partial \rho / \partial \beta)_p$ is determined from this equation of state using the formula, Eq. (70), in Lindblom and Mendell [30] that expresses this derivative in terms of the more easily evaluated derivatives of the chemical potentials. For the density range of primary interest to us $(2.8 \times 10^{14} \lesssim \rho \lesssim 10^{15}$ gm/cm^3), we find that

$$\left(\frac{\partial \rho}{\partial \beta} \right)_p \approx 1.4 \times 10^{-7} - 1.1 \times 10^{-22} \rho, \qquad (5.1)$$

where all quantities are expressed in cgs units. Similarly, we find that the proton fraction ρ_p / ρ for this equation of state is given approximately by

$$\frac{\rho_p}{\rho} \approx 0.031 + 8.8 \times 10^{-17} \rho. \qquad (5.2)$$

These expressions are accurate to within about 20% in the indicated density range.

The thermodynamic quantities $\tilde{\rho}$ and γ can be expressed in terms of the baryon densities ρ_n and ρ_p, and the mass-density matrix element ρ_{np} quite generally using the constraints on the mass-density matrix from Galilian invariance and Eq. (2.1):

$$\gamma = 1 + \epsilon \frac{\rho}{\rho_p}, \qquad (5.3)$$

$$\tilde{\rho} = \rho \left(1 - \frac{\rho_p}{\rho} \right) \left(\epsilon + \frac{\rho_p}{\rho} \right). \qquad (5.4)$$

Thus using Eq. (5.2) for the proton fraction, we obtain simple smooth expressions for the superfluid thermodynamic functions needed in Eqs. (4.18) and (4.19).

In order to evaluate the potentials δU_2 and $\delta \beta_2$, we convert the operators in Eqs. (4.18) and (4.19) into matrix operators by making the usual discrete representations of the various derivatives that appear. To this end we have given expressions for these operators in spherical coordinates in the Appendix. We convert the radial derivatives into discrete form using the standard three-point differencing formulas. For the angular derivatives, we have written separate codes that use either the higher-order angular differencing formulas discussed in Ipser and Lindblom [31] or the standard three-point formulas. We find that the results of our two separate codes agree very well. The code that uses the higher-order angular differencing formulas requires far fewer angular spokes in order to achieve a given accuracy, and it appears to be somewhat more stable.

The operator D that appears in Eq. (4.18) is a hyperbolic differential operator with a nontrivial kernel. Thus, Eq. (4.18) is solved using the numerical relaxation technique developed in Lindblom, Mendell, and Owen [33]. The stability and convergence properties of this technique were discussed at length in that work. The operator $\tilde{D}$ is also hyperbolic, but (generically) it does not have a nontrivial kernel. We find

that $\tilde{D}$ can be inverted numerically without difficulty. Thus, Eq. (4.19) can be solved by straightforward numerical techniques.

The Eqs. (4.18) and (4.19) are solved iteratively to determine the potentials δU_2 and $\delta \beta_2$. We begin the process by setting $\delta \beta_2 = 0$. Then we iterate the following sequence of calculations. First, we evaluate κ_2 from Eq. (4.24); then we solve Eq. (4.18) for δU_2; and finally we solve Eq. (4.19) for $\delta \beta_2$. This sequence of calculations is iterated until the functions δU_2 and $\delta \beta_2$ converge. This generally takes only a few steps. We find that the second-order eigenvalue κ_2 has the value 0.29879 for this model, essentially independent of the entrainment parameter ϵ and the superfluid-transition density ρ_s. This value of κ_2 differs only slightly from the value 0.29883 found by Lindblom, Mendell, and Owen [33] for the analogous ordinary-fluid r-modes.

The eigenfunction δU_2 for the case $\epsilon = 0.04$ and $\rho_s = 2.8 \times 10^{14}$ g/cm^3 is illustrated in Fig. 1. This function was computed on a grid having 2000 radial grid points and 10 angular spokes using the higher-order angular differencing code. Each curve represents the radial dependence of δU_2 along one of the angular spokes. This function is essentially identical to that obtained by Lindblom, Mendell, and Owen [33] for the ordinary-fluid case. Figure 2 illustrates the associated function $\delta \beta_2$ that was evaluated together with the δU_2 of Fig. 1. We note that $\delta \beta_2$ is only defined within the core of the star where the neutron-star matter is superfluid. We also note that $\delta \beta_2$ is about an order of magnitude smaller than δU_2. Thus even at the second-order, the superfluid r-modes differ little from their ordinary-fluid counterparts.

Figures 3 and 4 illustrate how $\delta \beta_2$ changes as ϵ varies. In particular Figs. 3 and 4 illustrate $\delta \beta_2$ for the extreme values of ϵ considered here: $\epsilon = 0.02$ and $\epsilon = 0.06$, respectively. While the differences in these three functions, Figs. 2, 3, and 4, are significant, they do not really illustrate the most interesting feature of their dependence on ϵ. The most interesting and unexpected feature that we find in the solutions for $\delta \beta_2$ is a kind of resonance phenomenon. We find that there are certain ''critical'' values of ϵ ($\epsilon_c = 0.02294$ and $\epsilon_c = 0.04817$ for the $\rho_s = 2.8 \times 10^{14}$ g/cm^3 case) near which the function $\delta \beta_2$ becomes extremely large. Near these special values of ϵ the character of the second-order terms in the expansion for the mode change from being dominated by the ordinary-fluid like correlated motion of the neutrons and protons, to the uniquely superfluid anti-correlated motion, where the neutrons and protons have opposite velocities. We see a hint of this behavior, perhaps, in Fig. 3 where the magnitude of $\delta \beta_2$ for $\epsilon = 0.02$ is considerably larger than its value at $\epsilon = 0.04$ or 0.06.

Mathematically we find that the reason for this resonance phenomenon is that the operator $\tilde{D}$ that determines $\delta \beta_2$ in Eq. (4.19) becomes singular at these critical values of ϵ. This singular behavior is illustrated in Fig. 5, in which we depict the dependence of the smallest (in absolute value) eigenvalue of the operator $\tilde{D}$ as a function of ϵ. We see that this smallest eigenvalue vanishes for the two critical values of ϵ noted above: $\epsilon_c = 0.02294$ and $\epsilon_c = 0.04817$. In a sufficiently small neighborhood of these points the function $\delta \beta_2$ must be pro-
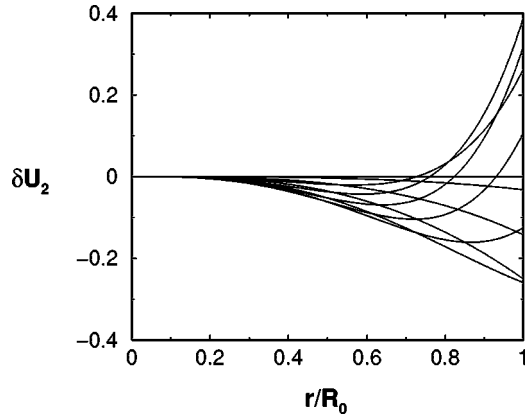
FIG. 1. Eigenfunction δU_2 for the $m=2$ r-mode based on a superfluid model with $\epsilon=0.04$ and $\rho_s=2.8\times10^{14}$ g/cm^3.
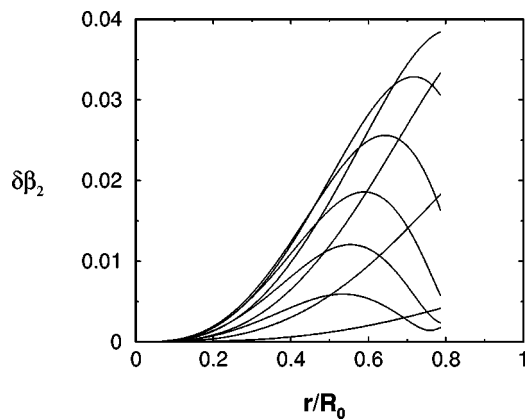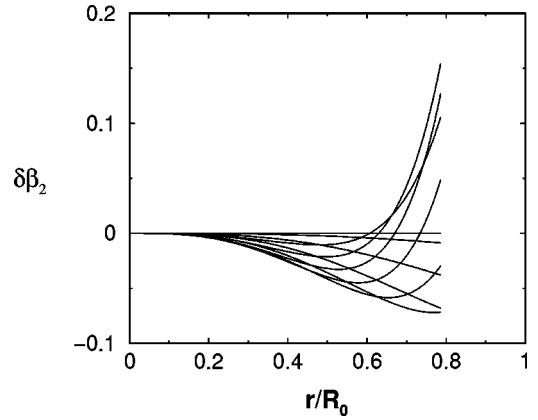

FIG. 3. Eigenfunction $\delta\beta_2$ for the $m=2$ r-mode based on a superfluid model with $\epsilon=0.02$ and $\rho_s=2.8\times10^{14}$ g/cm^3.

portional therefore to ψ_{min}/λ_{min} where ψ_{min} is the eigenfunction of $\tilde{D}$ that corresponds to this smallest eigenvalue, λ_{min}. Since λ_{min} vanishes for the singular values of ϵ, this forces the function $\delta\beta_2$ to become very large in sufficiently small neighborhoods of these points.

To paint a more physical picture of this resonance phenomenon, we note that the pulsation Eqs. (4.18) and (4.19) are analogous to the equations that describe a pair of coupled driven oscillators. The operators D and $\tilde{D}$ are analogous to the kinetic and potential terms of these oscillators; E and $\tilde{E}$ are the coupling between the oscillators; and F and $\tilde{F}$ are the driving terms. As the parameter ϵ changes, the operator $\tilde{D}$ changes while the other terms remain unchanged in these equations. Thus, varying ϵ is analogous to changing the spring constant of one of the oscillators while keeping the other oscillator and the coupling and driving terms fixed. In the case of coupled driven oscillators we know that the amplitude of one oscillator may be made large compared to the other if its spring constant is tuned to the appropriate critical value. This ''resonance'' behavior in coupled oscillators is analogous to the enhancement of $\delta\beta_2$ relative to δU_2 as ϵ is varied through one of its critical points. The modes in which $\delta\beta_2$ dominates δU_2 are somewhat analogous, therefore, to

the superfluid dominated s-modes discovered by Lindblom and Mendell [30], and studied subsequently by Lee [36] and by Comer, Langlois, and Lin [37]. The analogy is only superficial however. In the r-modes the fluid motion is always dominated by the potential δU_0, since the effects of $\delta\beta_2$ only enter at higher order in Ω. Further, the frequencies of the r-modes vanish in the limit of nonrotating stars, while the frequencies of the s-modes do not.

VI. DISSIPATION IN SUPERFLUID r-MODES

The primary motivation for our study here is to investigate how the transition to a superfluid state at low temperatures effects the stability of the r-modes in older colder neutron stars. The r-modes are driven towards instability by gravitational radiation [1,2], but internal fluid dissipation tends to suppress this instability [3,4]. Internal fluid dissipation can stabilize the r-modes completely, if it is sufficiently strong. In this section we investigate the importance of the two types of internal fluid dissipation considered most likely to have a substantial influence on the r-modes in superfluid neutron stars: mutual friction caused by the scattering of electrons off the cores of the neutron vortices, and shear viscosity due to electron-electron scattering.

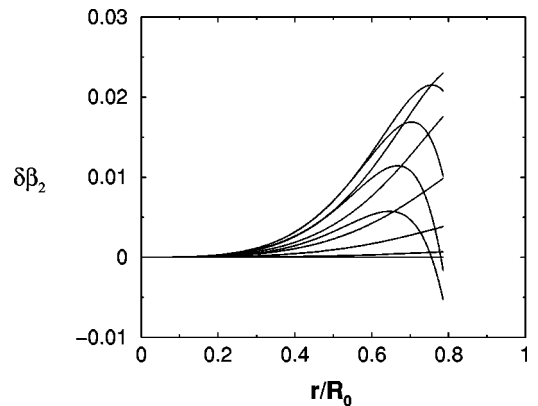The effects of dissipation on the evolution of a fluid are

FIG. 2. Eigenfunction $\delta\beta_2$ for the $m=2$ r-mode based on a superfluid model with $\epsilon=0.04$ and $\rho_s=2.8\times10^{14}$ g/cm^3.

FIG. 4. Eigenfunction $\delta\beta_2$ for the $m=2$ r-mode based on a superfluid model with $\epsilon=0.06$ and $\rho_s=2.8\times10^{14}$ g/cm^3.
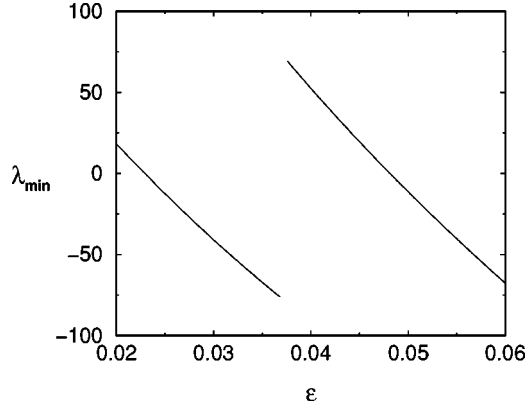
FIG. 5. Eigenvalue of the operator $\tilde{D}$ having the smallest absolute value as a function of ϵ, for $m=2$ perturbations with $\rho_s=2.8 \times 10^{14}$ g/cm^3.

most conveniently studied using an appropriate energy functional. For the case of a neutron-star superfluid whose evolution is determined by Eqs. (2.5), (2.6), (2.11), (2.12), and (2.13), the following is the appropriate energy functional [38]

$$\mathcal{E} = \frac{1}{2} \int \left\{ \rho \, \delta v_a^* \delta v^a + \rho \left(\frac{\partial \rho}{\partial p} \right)_\beta \text{Re}[(\delta U + \delta \Phi) \delta U^*] \right.$$

$$+ \tilde{\rho} \, \delta w_a^* \delta w^a + \frac{\rho_n^2}{\rho} \frac{\partial}{\partial \beta} \left(\frac{\rho_p}{\rho_n} \right)_p \delta \beta^* \delta \beta$$

$$\left. + \left(\frac{\partial \rho}{\partial \beta} \right)_p \text{Re}[(2 \delta U + \delta \Phi) \delta \beta^*] \right\} d^3x. \qquad (6.1)$$

The integrals of the first two terms on the right-hand side of Eq. (6.1) are to be performed throughout the star, while the last three terms (those proportional to δw^a or $\delta \beta$) are to be integrated only within the superfluid core. In the absence of dissipation, the energy $\mathcal{E}$ is conserved. When we evaluate the small angular-velocity expansion of the energy in Eq. (6.1), we find that only the first term on the right contributes at the lowest order. Thus, the lowest-order expression for the energy,

$$\mathcal{E} = \alpha^2 \frac{\pi(m+1)^3}{2m} (2m+1)!$$

$$\times R_0^4 \Omega^2 \int_0^{R_0} \rho \left(\frac{r}{R_0} \right)^{2m+2} dr + \mathcal{O}(\Omega^4), \qquad (6.2)$$

is identical to its ordinary-fluid counterpart [33].

Mutual friction tends to damp out any relative motion between the proton and neutron superfluids. This dissipation is caused by the scattering of electrons (whose motion tracks that of the proton superfluid in our approximation) off the magnetic fields that are entrapped within the cores of the neutron vortices. The presence of these magnetic fields is due to the entrainment effect, and hence the magnitude of this dissipation mechanism depends strongly on the poorly known mass-matrix element ρ_{np}. The rate at which energy is dissipated by mutual friction in the neutron-star superfluid considered here is given by [38]

$$\left(\frac{d\mathcal{E}}{dt} \right)_{MF} = -2\Omega \int B_n \rho_n \gamma^2 (\delta^{ab} - z^a z^b) \delta w_a \delta w_b^* d^3x, \qquad (6.3)$$

where the dimensionless mutual friction scattering coefficient, B_n, is given by,

$$B_n \approx \frac{\epsilon^2 \rho_p^{1/6}}{1.96 \times 10^4} \left(\frac{\rho}{\rho_p} - 1 \right) \left(1 - \epsilon + \epsilon \frac{\rho}{\rho_p} \right)^{-3/2}. \qquad (6.4)$$

We note that unlike regular viscosity, the mutual friction scattering coefficient does not depend on temperature (at least at the lowest order). This independence is due to the fact that mutual friction involves the scattering of two different fluids in relative motion. Thus, even at zero temperature some electrons acquire sufficient energy from their collisions with the vortices to scatter into available energy states located above the Fermi surface.

In order to evaluate the effects of mutual friction on the superfluid *r*-modes, it is necessary to re-express Eq. (6.3) in terms of the eigenfunction $\delta \beta_2$ that determines the relative superfluid motion. To lowest order in the angular velocity then, we find

$$\left(\frac{d\mathcal{E}}{dt} \right)_{MF} = -2R_0^4 (\pi G \bar{\rho}_0)^{3/2} \left(\frac{\Omega}{\sqrt{\pi G \bar{\rho}_0}} \right)^7 \int \frac{B_n \rho_0 \gamma_0^2 (1-\mu^2)}{(\kappa_0^2 - 4\gamma_0^2)^2} \left(1 - \frac{\rho_p}{\rho} \right)_0 \left[\left| \kappa_0 \left(\frac{\partial \delta \beta_2}{\partial r} - \frac{\mu}{r} \frac{\partial \delta \beta_2}{\partial \mu} \right) + \frac{2m \gamma_0 \delta \beta_2}{r(1-\mu^2)} \right|^2 \right.$$

$$\left. + \left| 2\gamma_0 \left(\frac{\partial \delta \beta_2}{\partial r} - \frac{\mu}{r} \frac{\partial \delta \beta_2}{\partial \mu} \right) + \frac{m \kappa_0 \delta \beta_2}{r(1-\mu^2)} \right|^2 \right] d^3x. \qquad (6.5)$$

It is straightforward to perform these integrals numerically using the superfluid *r*-mode eigenfunctions described in Sec. V, and so determine the energy dissipation rate caused by mutual friction. It is convenient to express this rate in terms of a mutual-friction damping time, τ_{MF}:
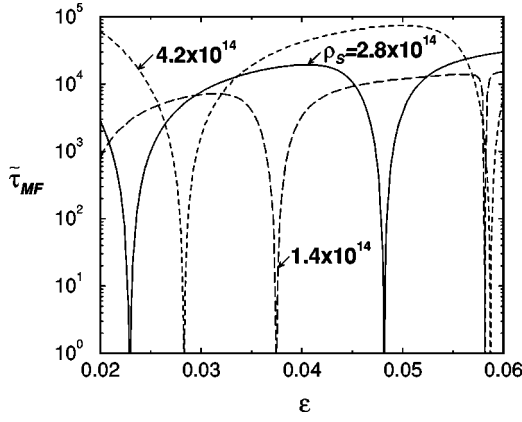
FIG. 6. Characteristic damping time $\tilde{\tau}_{MF}$ due to superfluid mutual friction. These curves show the dependence of $\tilde{\tau}_{MF}$ as a function of the superfluid entrainment parameter ϵ for three values of the superfluid transition density ρ_s.

$$\frac{1}{\tau_{MF}} = -\frac{1}{2\mathcal{E}}\left(\frac{d\mathcal{E}}{dt}\right)_{MF} = \frac{1}{\tilde{\tau}_{MF}}\left(\frac{\Omega}{\sqrt{\pi G \bar{\rho}_0}}\right)^5. \quad (6.6)$$

The characteristic mutual-friction damping time $\tilde{\tau}_{MF}$, also defined in Eq. (6.6), is independent of angular velocity and temperature (to lowest order). The Ω^5 scaling of $1/\tau_{MF}$ follows directly from Eqs. (6.2) and (6.5). The Ω^7 scaling of $d\mathcal{E}/dt$ in Eq. (6.5) follows in turn from Eq. (6.3) and the fact that δw^a scales as Ω^3 to lowest order. We present in Figs. 6 and 7 the numerically determined characteristic damping times $\tilde{\tau}_{MF}$ for the $m=2$ superfluid r-modes. Figure 6 illustrates how sensitively $\tilde{\tau}_{MF}$ depends on the entrainment parameter ϵ. We see that $\tilde{\tau}_{MF}$ has a typical value of about 10^4 s, but that it becomes *much* smaller for a few narrow ranges of ϵ. These spikes in the $\tilde{\tau}_{MF}(\epsilon)$ curves are caused by the ''resonance'' phenomenon that we discuss in Sec. V. The mutual friction damping time $\tilde{\tau}_{MF}$ becomes very small when the eigenfunction $\delta\beta_2$ becomes large. This occurs at the points where the operator $\tilde{D}$ has a vanishing eigenvalue. We see from Fig. 5 that the locations of these vanishing eigenvalues coincide exactly with the location of the spikes in the $\rho_s = 2.8 \times 10^{14}$ g/cm^3 curve in Fig. 6. We find that near these spikes the curve $\tilde{\tau}_{MF}(\epsilon) \propto (\epsilon - \epsilon_c)^2$. This quadratic dependence is exactly what is expected given that $\delta\beta_2 \propto \psi_{\min}/\lambda_{\min}$ near these spikes. Figure 6 also illustrates that the exact location of these spikes depends on the value of the superfluid transition density ρ_s. Figure 7 illustrates that $\tilde{\tau}_{MF}$ changes smoothly as ρ_s is varied: the locations of these spikes move smoothly to larger values of ϵ as ρ_s increases. Figure 6 also illustrates that the probability of having a small $\tilde{\tau}_{MF}$ does not depend strongly on ρ_s. For example, we find that about 1% of the values of ϵ in the acceptable range, $0.02 \leqslant \epsilon \leqslant 0.06$, have $\tilde{\tau}_{MF}(\epsilon) \leqslant 5$ s, and this percentage is relatively insensitive to ρ_s.

The other important internal fluid dissipation mechanism in superfluid neutron stars is expected to be regular shear

viscosity. In the superfluid core this viscosity is due to electron-electron scattering, while in the surrounding ordinary-fluid envelope the standard neutron-neutron scattering dominates. The rate at which energy is dissipated by shear viscosity is given by

$$\left(\frac{d\mathcal{E}}{dt}\right)_V = -\int_V 2\eta\delta\sigma_{ab}^* \delta\sigma^{ab} d^3x + \int_s 2\eta n^a \delta v_e^b \delta\sigma_{ab}^* d^2x$$
$$-\int_o 2\eta n^a \delta v^b \delta\sigma_{ab}^* d^2x. \quad (6.7)$$

The volume integral in Eq. (6.7) is to be evaluated within the superfluid core, and within the ordinary-fluid envelope, but not over the boundary surface between the two. The surface integrals are to be evaluated over the interior (superfluid side) and exterior (ordinary-fluid side) of the superfluid boundary, respectively. The tensor $\delta\sigma_{ab}$ that appears in these integrals is the shear of the electron velocity δv_e^a of Eq. (2.4) within the superfluid core, and the ordinary-fluid velocity δv^a in the envelope. In the small angular velocity expansion the electron velocity δv_e^a is the same as δv^a, to lowest order. Thus to lowest order the energy dissipation rate due to shear viscosity is given by

$$\left(\frac{d\mathcal{E}}{dt}\right)_V = -\alpha^2 \frac{\pi(m+1)^3}{m}(m-1)(2m+1)! R_0^2 \Omega^2$$
$$\times \left\{ (2m+1)\int_0^{R_0} \eta\left(\frac{r}{R_0}\right)^{2m} dr \right.$$
$$\left. -(\eta_s - \eta_o)R_s\left(\frac{R_s}{R_0}\right)^{2m} \right\}, \quad (6.8)$$

where η_s and η_o are the limits of the viscosity taken from the superfluid and the ordinary-fluid side of the boundary, respectively.

In the superfluid core of the neutron star, $r \leqslant R_s$, the appropriate viscosity to use in Eq. (6.8) is due to electron-
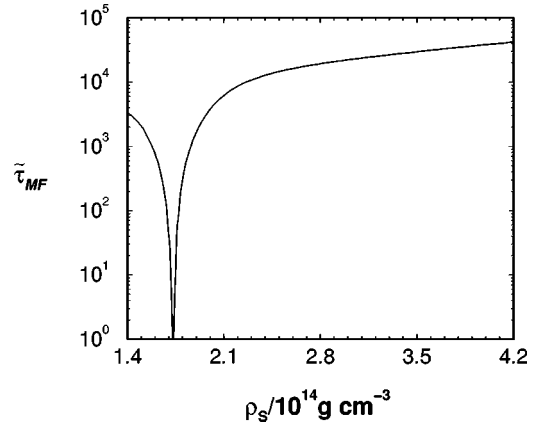

FIG. 7. Characteristic damping time $\tilde{\tau}_{MF}$ due to superfluid mutual friction. This curve shows the dependence of $\tilde{\tau}_{MF}$ as a function of the superfluid transition density ρ_s for $\epsilon = 0.04$.

electron scattering. This electron-electron scattering viscosity is given approximately by the expression [39]

$$\eta = 6.0 \times 10^6 \left(\frac{\rho}{T} \right)^2,$$ (6.9)

where all quantities are given in cgs units. Similarly, in the ordinary-fluid envelope, neutron-neutron scattering viscosity dominates at the densities where most of the dissipation occurs. This neutron-neutron scattering viscosity is given approximately by the expression [39]

$$\eta = 347 \frac{\rho^{9/4}}{T^2}.$$ (6.10)

Using these expressions for the viscosity it is straightforward to evaluate the energy dissipation rate for shear viscosity using Eq. (6.8). As in the case of mutual friction, it is convenient to express the result as a viscous time scale:

$$\frac{1}{\tau_V} = -\frac{1}{2\mathcal{E}} \left(\frac{d\mathcal{E}}{dt} \right)_V = \frac{1}{\tilde{\tau}_V} \left(\frac{10^9 \ \mathrm{K}}{T} \right)^2.$$ (6.11)

The characteristic viscous time scale $\tilde{\tau}_V$, also defined in Eq. (6.11), is independent of the angular velocity and the temperature of the neutron star. We find that $\tilde{\tau}_V = 1.01 \times 10^8$ s for our superfluid neutron-star model with $\rho_s = 2.8 \times 10^{14}$ g/cm^3. This value is somewhat shorter than that obtained for hot neutron stars, 2.52×10^8 s, by Lindblom, Owen, and Morsink [3] using neutron-neutron scattering viscosity [Eq. (6.10)] throughout the star; and for cold neutron stars, 2.25×10^8 s, by Andersson, Kokkotas, and Schutz [4] using electron-electron scattering viscosity [Eq. (6.9)] throughout.

Gravitational radiation is the final form of dissipation expected to have a significant influence on the *r*-modes of superfluid neutron stars. Since the superfluid *r*-modes are identical to their ordinary-fluid counterparts to lowest order in the angular velocity, the gravitational radiation coupling is the same to this order. The characteristic gravitational radiation time scale $\tilde{\tau}_{GR}$ for the $m = 2$ *r*-mode,

$$\frac{1}{\tau_{GR}} = -\frac{1}{2\mathcal{E}} \left(\frac{d\mathcal{E}}{dt} \right)_{GR} = -\frac{1}{\tilde{\tau}_{GR}} \left(\frac{\Omega}{\sqrt{\pi G \bar{\rho}_0}} \right)^6, \quad (6.12)$$

therefore has the same value, $\tilde{\tau}_{GR} = 3.26$ s, as in the ordinary-fluid case [3].

The effects of mutual friction, shear viscosity, and gravitational radiation act together simultaneously to influence the evolution of the superfluid *r*-modes. Their combined effects on the evolution of the energy of the mode are conveniently described by the overall dissipative time scale, τ:

$$\frac{1}{\tau} = -\frac{1}{2\mathcal{E}} \left\{ \left(\frac{d\mathcal{E}}{dt} \right)_{GR} + \left(\frac{d\mathcal{E}}{dt} \right)_{MF} + \left(\frac{d\mathcal{E}}{dt} \right)_V \right\}. \quad (6.13)$$

Since the energy $\mathcal{E}$ is a real functional, the quantity $1/\tau$ is in fact just the imaginary part of the frequency of the mode. The sign of τ therefore determines whether the mode is stable. If $\tau > 0$ then dissipation decreases the energy of the mode and it is stable, while if $\tau < 0$ then dissipation causes the energy (and hence the mode itself) to increase exponentially. We can explore the conditions under which mutual friction and viscosity are effective in suppressing the gravitational radiation driven instability by giving the explicit temperature and angular-velocity dependence of τ using Eqs. (6.6), (6.11), and (6.12):

$$\frac{1}{\tau(\Omega, T)} = -\frac{1}{\tilde{\tau}_{GR}} \left(\frac{\Omega}{\sqrt{\pi G \bar{\rho}_0}} \right)^6 + \frac{1}{\tilde{\tau}_{MF}} \left(\frac{\Omega}{\sqrt{\pi G \bar{\rho}_0}} \right)^5$$

$$+ \frac{1}{\tilde{\tau}_V} \left(\frac{10^9 \ \mathrm{K}}{T} \right)^2.$$ (6.14)

Gravitational radiation tends to drive the *r*-modes unstable while mutual friction and shear viscosity tend to stabilize these modes. From Eq. (6.14) we see that $\tau > 0$, and mutual friction will completely suppress the gravitational radiation driven instability whenever

$$\tilde{\tau}_{MF} \left(\frac{\Omega}{\sqrt{\pi G \bar{\rho}_0}} \right) \le \tilde{\tau}_{GR}.$$ (6.15)

Since neutron stars have angular velocities that are limited by $\Omega \lesssim \frac{2}{3} \sqrt{\pi G \bar{\rho}_0}$, we see that mutual friction will suppress the gravitational radiation instability for all angular velocities whenever $\tilde{\tau}_{MF} \lesssim 1.5 \tilde{\tau}_{GR} \approx 4.89$ s. From Fig. 6 we see that this may occur, but only if the entrainment parameter ϵ of neutron-star matter is limited to a very narrow range. Only about 1% of the values of ϵ in the expected range, $0.02 \le \epsilon \le 0.06$, have sufficiently short $\tilde{\tau}_{MF}$ to suppress the instability completely.

From Eq. (6.15) we see that mutual friction will always suppress the gravitational radiation driven instability in the *r*-modes of neutron stars with sufficiently small angular ve-
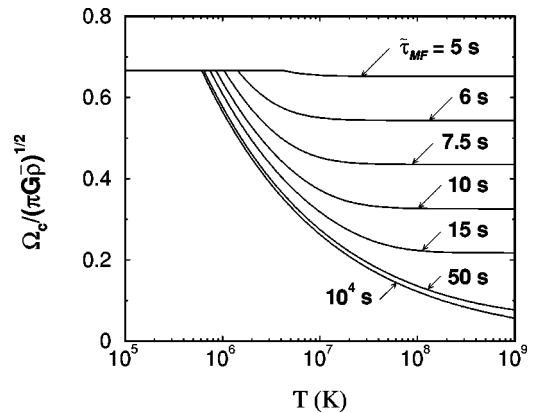

FIG. 8. Critical angular velocities for superfluid neutron stars having a range of characteristic mutual friction time scales $\tilde{\tau}_{MF}$.

locities. However, this suppression is not physically relevant if $\tilde{\tau}_{MF}$ is too large. If the angular velocities needed in Eq. (6.15) are sufficiently small, then shear viscosity will also play a significant role in suppressing the r-mode instability in these stars. In particular shear viscosity alone will suppress the r-mode instability if

$$\frac{\Omega}{\sqrt{\pi G \bar{\rho}_0}} \lesssim \left(\frac{\tilde{\tau}_{GR}}{\tilde{\tau}_V}\right)^{1/6} \left(\frac{10^9 \ \text{K}}{T}\right)^{1/3} \approx 0.0564 \left(\frac{10^9 \ \text{K}}{T}\right)^{1/3}. \tag{6.16}$$

Mutual friction then is capable of further suppressing the r-mode instability in more rapidly rotating neutron stars only if

$$\tilde{\tau}_{MF} \lesssim \tilde{\tau}_{GR} \left(\frac{\tilde{\tau}_V}{\tilde{\tau}_{GR}}\right)^{1/6} \left(\frac{T}{10^9 \ \text{K}}\right)^{1/3} \approx 57.7 \ \text{s} \left(\frac{T}{10^9 \ \text{K}}\right)^{1/3}. \tag{6.17}$$

We see that mutual friction will be primarily responsible for the suppression of the r-mode instability in some (sufficiently warm) superfluid neutron stars only if $\tilde{\tau}_{MF} \lesssim 57.7$ s. Such small values of $\tilde{\tau}_{MF}$ occur in only about 3% of the values of ϵ in the expected range: $0.02 \leq \epsilon \leq 0.06$.

It is also informative to examine the critical angular velocities, Ω_c, that mark the dividing line between stable and unstable neutron stars: $\tau(\Omega_c, T) = 0$. Stars rotating more rapidly than Ω_c are unstable, while those rotating more slowly are stable. Figure 8 illustrates the temperature dependence of Ω_c for a range of possible mutual friction time-scales. From Fig. 8 we see that shear viscosity completely suppresses the gravitational radiation instability in all neutron stars cooler than about 10^6 K. We also see that the mutual friction time scale must be shorter than 10 or 15 s in order for mutual friction to play a significant role in suppressing the gravitational radiation instability in neutron stars with temperatures that are typical of low mass x-ray binaries (about 10^8 K). Since only about 2% of the expected range of ϵ have time scales this short, it appears unlikely that mutual friction is acting to suppress the gravitational radiation instability in these stars.

ACKNOWLEDGMENTS

APPENDIX: PULSATION EQUATIONS IN SPHERICAL COORDINATES

In order to solve the superfluid pulsation equations numerically, it is necessary to express them in some particular coordinate representation. We find it useful to work in spherical coordinate: r, $\mu = \cos \theta$, and φ. The operators D, E, F, $\tilde{D}$, $\tilde{E}$, and $\tilde{F}$ that appear in Eqs. (4.18) and (4.19) have the following representations in spherical coordinates:

$$\begin{aligned} D(\delta U_2) = \kappa_0^2 \, \rho_0 &\left[\frac{\partial^2 \delta U_2}{\partial r^2} + \frac{1-\mu^2}{r^2} \frac{\partial^2 \delta U_2}{\partial \mu^2} + \frac{2}{r} \frac{\partial \delta U_2}{\partial r} - \frac{2\mu}{r^2} \frac{\partial \delta U_2}{\partial \mu} - \frac{m^2 \delta U_2}{r^2(1-\mu^2)} \right] \\ &- 4\rho_0 \left[\frac{2\mu(1-\mu^2)}{r} \frac{\partial^2 \delta U_2}{\partial r \partial \mu} + \mu^2 \frac{\partial^2 \delta U_2}{\partial r^2} + \frac{(1-\mu^2)^2}{r^2} \frac{\partial^2 \delta U_2}{\partial \mu^2} + \frac{1-\mu^2}{r} \frac{\partial \delta U_2}{\partial r} - \frac{3\mu(1-\mu^2)}{r^2} \frac{\partial \delta U_2}{\partial \mu} \right] \\ &+ \frac{d\rho_0}{dr} \left[(\kappa_0^2 - 4\mu^2) \frac{\partial \delta U_2}{\partial r} + \frac{2m\kappa_0}{r} \delta U_2 - \frac{4\mu(1-\mu^2)}{r} \frac{\partial \delta U_2}{\partial \mu} \right], \end{aligned} \tag{A1}$$

$$\begin{aligned} E(\beta_2) = (\kappa_0^2 - 4\mu^2) r \frac{d}{dr} &\left[\frac{1}{r} \frac{dh_0}{dr} \left(\frac{\partial \rho}{\partial \beta} \right)_p \right] \delta\beta_2 + \frac{dh_0}{dr} \left(\frac{\partial \rho}{\partial \beta} \right)_p \left[(\kappa_0^2 - 4\mu^2) \frac{\partial \delta\beta_2}{\partial r} - \frac{4\mu(1-\mu^2)}{r} \frac{\partial \delta\beta_2}{\partial \mu} \right] \\ &+ (3\kappa_0^2 - 2m\kappa_0 - 4) \frac{1}{r} \frac{dh_0}{dr} \left(\frac{\partial \rho}{\partial \beta} \right)_p \delta\beta_2, \end{aligned} \tag{A2}$$

$$F = \frac{12m(m+2)}{(m+1)^2} \frac{\rho_{22}}{r^2} \delta U_0 - 2(m+2)\kappa_2 \frac{1}{r} \frac{d\rho_0}{dr} \delta U_0 + 16\pi G \bar{\rho}_0 \frac{m(m+2)}{(m+1)^4} \left(\frac{d\rho}{dh} \right)_0 (\delta U_0 + \delta\Phi_0), \tag{A3}$$

$$\widetilde{D}(\delta\beta_2) = \frac{\kappa_0^2\,\widetilde{\rho}_0}{\kappa_0^2 - 4\,\gamma_0^2}\left[\frac{\partial^2\delta\beta_2}{\partial r^2} + \frac{1-\mu^2}{r^2}\frac{\partial^2\delta\beta_2}{\partial\mu^2} + \frac{2}{r}\frac{\partial\delta\beta_2}{\partial r} - \frac{2\mu}{r^2}\frac{\partial\delta\beta_2}{\partial\mu} - \frac{m^2\delta\beta_2}{r^2(1-\mu^2)}\right]$$

$$- \frac{4\,\gamma_0^2\,\widetilde{\rho}_0}{\kappa_0^2 - 4\,\gamma_0^2}\left[\mu^2\frac{\partial^2\delta\beta_2}{\partial r^2} + \frac{2\mu(1-\mu^2)}{r}\frac{\partial^2\delta\beta_2}{\partial r\,\partial\mu} + \frac{(1-\mu^2)^2}{r^2}\frac{\partial^2\delta\beta_2}{\partial\mu^2} + \frac{1-\mu^2}{r}\frac{\partial\delta\beta_2}{\partial r} - \frac{3\mu(1-\mu^2)}{r^2}\frac{\partial\delta\beta_2}{\partial\mu}\right]$$

$$+ \frac{d}{dr}\left(\frac{\kappa_0^2\,\widetilde{\rho}_0}{\kappa_0^2 - 4\,\gamma_0^2}\right)\frac{\partial\delta\beta_2}{\partial r} - \frac{d}{dr}\left(\frac{4\,\gamma_0^2\,\widetilde{\rho}_0}{\kappa_0^2 - 4\,\gamma_0^2}\right)\left[\mu^2\frac{\partial\delta\beta_2}{\partial r} + \frac{\mu(1-\mu^2)}{r}\frac{\partial\delta\beta_2}{\partial\mu}\right]$$

$$- \frac{\kappa_0^2 - 4\mu^2}{\rho_0(\kappa_0^2 - 4)}\left(\frac{\partial\rho}{\partial\beta}\right)_p^2\left(\frac{dh_0}{dr}\right)^2\delta\beta_2 + \frac{2m\kappa_0}{r}\frac{d}{dr}\left(\frac{\gamma_0\,\widetilde{\rho}_0}{\kappa_0^2 - 4\,\gamma_0^2}\right)\delta\beta_2, \tag{A4}$$

$$\widetilde{E}(\delta U_2) = -\frac{1}{\kappa_0^2 - 4}\frac{dh_0}{dr}\left(\frac{\partial\rho}{\partial\beta}\right)_p\left[(\kappa_0^2 - 4\mu^2)\frac{\partial\delta U_2}{\partial r} - \frac{4\mu(1-\mu^2)}{r}\frac{\partial\delta U_2}{\partial\mu} + \frac{2m\kappa_0}{r}\delta U_2\right], \tag{A5}$$

$$\widetilde{F} = \left(\frac{\partial\rho}{\partial\beta}\right)_p\left[\frac{(m+1)^2\kappa_2}{2mr}\frac{dh_0}{dr} - \frac{3h_{22}}{r^2}\right]\delta U_0 - \frac{4\pi G\overline{\rho}_0}{(m+1)^2}\left(\frac{\partial\rho}{\partial\beta}\right)_p(\delta U_0 + \delta\Phi_0). \tag{A6}$$

We note that the functions ρ_{22} and h_{22} that appear on the right-hand sides of Eqs. (A3) and (A6) are parts of the second-order expansions of these quantities. In general the second-order density function has the form: $\rho_2(r,\mu) = \rho_{20}(r) + \rho_{22}(r)P_2(\mu)$, where $\rho_{20}(r)$ and $\rho_{22}(r)$ are functions of r and $P_2(\mu)$ is the $l=2$ Legendre polynomial. A similar expression is satisfied by h_2. These functions may be determined using the techniques described in Lindblom, Mendell, and Owen [33].

[1] N. Andersson, Astrophys. J. **502**, 708 (1998).
[2] J. L. Friedman and S. M. Morsink, Astrophys. J. **502**, 714 (1998).
[3] L. Lindblom, B. J. Owen, and S. M. Morsink, Phys. Rev. Lett. **80**, 4843 (1998).
[4] N. Andersson, K. Kokkotas, and B. F. Schutz, Astrophys. J. **510**, 846 (1999).
[5] B. J. Owen, L. Lindblom, C. Cutler, B. F. Schutz, A. Vecchio, and N. Andersson, Phys. Rev. D **58**, 084020 (1998).
[6] N. Andersson, K. D. Kokkotas, and N. Stergioulas, Astrophys. J. **516**, 307 (1999).
[7] L. Bildsten, Astrophys. J. Lett. **501**, L89 (1998).
[8] Y. Levin, Astrophys. J. **517**, 328 (1999).
[9] D. C. Backer, S. R. Kulkarni, C. Heiles, M. M. Davis, and W. M. Goss, Nature (London) **300**, 615 (1980); A. S. Fruchter, D. R. Stinebring, and J. H. Taylor, *ibid.* **333**, 237 (1988).
[10] D. C. Backer, Astrophys. J. **493**, 873 (1998).
[11] G. B. Cook, S. L. Shapiro, and S. A. Teukolsky, Astrophys. J. **424**, 823 (1994).
[12] E. S. Phinney and S. R. Kulkarni, Annu. Rev. Astron. Astrophys. **32**, 591 (1994).
[13] M. A. Alpar, S. A. Langer, and J. A. Sauls, Astrophys. J. **282**, 533 (1984).
[14] M. A. Alpar and J. A. Sauls, Astrophys. J. **327**, 725 (1988).
[15] L. Lindblom and G. Mendell, Astrophys. J. **444**, 804 (1995).
[16] E. F. Brown and L. Bildsten, Astrophys. J. **496**, 915 (1998).
[17] G. Baym and C. J. Pethick, Annu. Rev. Nucl. Sci. **25**, 27 (1975); Annu. Rev. Astron. Astrophys. **17**, 415 (1979).
[18] D. Pines and M. A. Alpar, Nature (London) **316**, 27 (1985).
[19] J. A. Sauls, in *Timing Neutron Stars*, edited by H. Ögelman

and E. P. J. van den Heuvel (Kluwer, Dordrecht, 1988), p. 457.
[20] R. I. Epstein, Astrophys. J. **333**, 880 (1988).
[21] I. L. Bekarevich and I. M. Khalatnikov, Zh. Éksp. Teor. Fiz. **40**, 920 (1961) [Sov. Phys. JETP **13**, 643 (1961)].
[22] G. Baym and E. Chandler, J. Low Temp. Phys. **50**, 57 (1983).
[23] E. B. Sonin, Rev. Mod. Phys. **59**, 87 (1987).
[24] G. Mendell and L. Lindblom, Ann. Phys. (N.Y.) **205**, 110 (1991).
[25] G. Mendell, Astrophys. J. **380**, 515 (1991).
[26] G. Mendell, Mon. Not. R. Astron. Soc. **296**, 903 (1998).
[27] A. F. Andreev and E. P. Bashkin, Zh. Éksp. Teor. Fiz. **69**, 319 (1975) [Sov. Phys. JETP **42**, 164 (1976)].
[28] B. Carter and D. Langlois, Nucl. Phys. **B454**, 402 (1998); **B531**, 478 (1998).
[29] M. Borumand, R. Joynt, and W. Kluźniak, Phys. Rev. C **54**, 2745 (1996).
[30] L. Lindblom and G. Mendell, Astrophys. J. **421**, 689 (1994).
[31] J. R. Ipser and L. Lindblom, Astrophys. J. **355**, 226 (1990).
[32] J. Papaloizou and J. E. Pringle, Mon. Not. R. Astron. Soc. **182**, 423 (1978).
[33] L. Lindblom, G. Mendell, and B. J. Owen, Phys. Rev. D **60**, 064006 (1999).
[34] L. Lindblom and J. R. Ipser, Phys. Rev. D **59**, 044009 (1999).
[35] A. Akmal, V. R. Pandharipande, and D. G. Ravenhall, Phys. Rev. C **58**, 1804 (1998).
[36] U. Lee, Astron. Astrophys. **303**, 515 (1995).
[37] G. L. Comer, D. Langlois, and L. M. Lin, Phys. Rev. D **60**, 104025 (1999).
[38] G. Mendell, Astrophys. J. **380**, 530 (1991).
[39] C. Cutler and L. Lindblom, Astrophys. J. **314**, 234 (1987).